ANALYSIS III FINAL EXAMINATION

Total marks: 100
Attempt all questions
Time: 3 hours

(1) Let $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be a C^1 function, and let $a \in \mathbb{R}^{n+k}$ with $f(a) = 0$. Assume that the total derivative $Df(a)$ of f at a has rank n. Prove that if c is a point of $\mathbb{R}^n$ sufficiently close to 0, then $f(x) = c$ has a solution. (15 marks)

(2) Give an example of a bounded open set which is not rectifiable. (15 marks)

(3) Use Green's theorem to compute the line integral $\int_C y^2 dx + x dy$, where C is the square with vertices $(0,0), (0,2), (2,2), (2,0)$. (10 marks)

(4) Compute the area of the portion of the surface $z^2 = 2xy$ which lies above the first quadrant of the xy-plane and is cut off by the planes $x = 2$ and $y = 1$. (20 marks)

(5) Let $S_n(a) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \text{ s.t.} |x_1| + \ldots + |x_n| \leq a\}$ (for any $a > 0$). Define the volume of $S_n(a)$ to be $V_n(a) = \int_{S_n(a)} 1$ (assume the integral exists). Prove that $V_n(a) = 2^n a^n / n!$. (20 marks)

(6) Let $F(x, y, z) = (y^2, xy, xz)$, let S be that part of the paraboloid surface $z = x^2 + y^2$ which lies in the cylinder $x^2 + y^2 = 1$, and consider the unit normal n on S with non-negative z component. Verify Stokes theorem by computing both sides of the equation in the statement. (20 marks)